arXiv:1303.6210v1 [math.AP] 25 Mar 2013

STEADY FLUID FLOW IN FISSURED POROUS MEDIA WITH IMPERFECT CONTACT

ABDELHAMID AINOUZ

ABSTRACT. In this paper, periodic homogenization of a steady fluid flow in fissured porous solids with imperfect interfacial contact is performed via two-scale asymptotic method.

1. INTRODUCTION

Micro-models for the fluid flow in fissured porous medium made of two interacting porous systems with different permeabilities are very important in reservoir petroleum, civil engineering, geophysics and many other areas of engineering. In this paper, we shall deal with the homogenization of fluid flow in fissured porous media. There are many papers devoted to the subject, see for instance [4, 5, 7] and the references therein. Here, we shall be concerned with periodic porous media made of two components with highly contrast of permeabilities and an imperfect contact on the interface of these solids. One of the two porous structures is associated with the fissures and the other one with the porous block or matrix. More precisely, we shall consider a Newtonian fluid flow, in a periodic fissured porous medium $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$, where Ω_1^ε is the fissures region, Ω_2^ε is the block and Γ^ε is the interface that separates these two regions. The micro-model is based on mass conservation for the fluid in each phase, combined with the Darcy's law . We assume that Ω_1^ε and Ω_2^ε are in imperfect contact. That is, an exchange flow barrier formulation on the interface Γ^ε will be considered, see for instance, [1, 8, 9]. The macro-model is derived by means of two-scale asymptotic technique (*Cf.* [6, 10]) and jusitified with the help of the two-scale convergence method [3]. It will be seen that the macro-model is, in some sense, the limit of a family of periodic micro-models in which the size of the periodicity approaches zero. It is shown that the overall behavior of fluid flow in such micro-scale heterogeneous media, is a standard one. Namely, the macroscopic equation is of elliptic type and it has the form $-\mathrm{div}\big(A^h \nabla u\big) = F$. The only novelty here is that the fluid flow presents an extra source surface density of the exterior boundary. We mention that no double porosity effects occur at the macroscale description.

The paper is organized as follows: Section 2 is devoted to the problem setting of the micro-model. In Section 3, we shall be concerned with the derivation of the homogenized model via the formal procedure of two-scale asymptotic ananlysis. Section 4 is devoted to the rigorous derivation of the homogenized model obtained in the previous section by the two scale convergence method. Finally, we end this paper with some comments and conclusion.

2000 *Mathematics Subject Classification.* Primary 35B27, 76S50.

Key words and phrases. Homogenization, fractured media.

The paper is a part of the project N° B00002278 of the MESRS algerian office. The author is indebted for their financial support.

2. Setting of the Problem and the main result

We consider Ω a bounded, connected and smooth open subset of $\mathbb{R}^N$ $(N \geq 2)$. We assume that Ω is made of a large number of identical cells εY, where $\varepsilon > 0$ is a sufficiently small parameter ($\varepsilon \ll 1$) and $Y =]0,1[^N$ denotes the generic cell of periodicity. Let Y be divided as follows: $Y := Y_1 \cup Y_2 \cup \Gamma$ where Y_1, Y_2 are two subdomains of Y and Γ is the interface between Y_1 and Y_2. Namely $\Gamma = \overline{Y_1} \cap \overline{Y_2}$. We denote ν the unit normal of Γ, outward to Y_1. Let χ_1 and χ_2 denote respectively the characteristic function of Y_1 and Y_2, extended by Y-periodicity to $\mathbb{R}^N$. For $i = 1$ or 2 we set

$$\Omega_\varepsilon^i := \{x \in \Omega : \chi_i(\frac{x}{\varepsilon}) = 1\}.$$

Let $\Gamma^\varepsilon := \overline{\Omega_1^\varepsilon} \cap \overline{\Omega_2^\varepsilon}$. We will assume that $\overline{\Omega_2^\varepsilon} \subset \Omega$, so that $\partial\Omega_2^\varepsilon = \Gamma^\varepsilon$ and $\partial\Omega_1^\varepsilon = \partial\Omega \cup \Gamma^\varepsilon$. Let $Z_i = \cup_{k\in\mathbb{Z}^N}(Y_i + k)$. As in [3], we shall assume that Z_1 is smooth and connected open subset of $\mathbb{R}^N$.

Let A and B denote respectively the permeability of the medium Z_1 and Z_2. We assume that A (resp. B) is continuous on $\mathbb{R}^N$, $Y-$periodic and satisfies the ellipticity condition:

$$A\xi \cdot \xi \geq C|\xi|^2, (\text{resp. } B\xi \cdot \xi \geq C|\xi|^2) \quad \xi \in \mathbb{R}^N$$

where, here and in what follows, C denotes various positive constants independent of ε. Let f_i be a measurable function representing the internal source density of the fluid flow in Ω_i^ε and let h be the *non-rescaled* hydraulic permeability of the thin layer Γ^ε. We suppose that $f_i \in L^2(\Omega)$ and that h is a continous function on $\mathbb{R}^N$, $Y-$periodic and bounded from below:

$$h(y) \geq h_0 > 0 \; x \in \mathbb{R}^N.$$

To deal with periodic homogenization with microstructures, we shall denote for $x \in \mathbb{R}^N$, $A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$, $B^\varepsilon(x) = B\left(\frac{x}{\varepsilon}\right)$ and $h^\varepsilon(x) = \varepsilon h\left(\frac{x}{\varepsilon}\right)$.

The micro-model that we shall study is given by the following set of equations:

$$-\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f_1 \text{ in } \Omega_1^\varepsilon, \tag{2.1a}$$

$$-\varepsilon^2 \operatorname{div}(B^\varepsilon \nabla v^\varepsilon) = f_2 \text{ in } \Omega_2^\varepsilon, \tag{2.1b}$$

$$A^\varepsilon \nabla u^\varepsilon \cdot \nu^\varepsilon = \varepsilon^2 B^\varepsilon \nabla v^\varepsilon \cdot \nu^\varepsilon \text{ on } \Gamma^\varepsilon, \tag{2.1c}$$

$$A^\varepsilon \nabla u^\varepsilon \cdot \nu^\varepsilon = -\varepsilon h^\varepsilon (u^\varepsilon - v^\varepsilon) \text{ on } \Gamma^\varepsilon, \tag{2.1d}$$

$$u^\varepsilon = 0 \text{ on } \partial\Omega \tag{2.1e}$$

where ν^ε stands for the unit normal of Γ^ε outward to Ω_1^ε. It is obtained by the $Y-$periodicity extension of ν. Let us mention that Ω_1^ε represents the fissured space region with permeability A and Ω_2^ε the block region with permeability $\varepsilon^2 B$. The quantitities u^ε and v^ε are the fluid flow velocities in Ω_1^ε and Ω_2^ε respectively. Note tha we have chosen a particular scaling of the permeability coefficients in (2.1b). This means that both terms $\int_{\Omega_1^\varepsilon} |\nabla u^\varepsilon|^2 dx$ and $\varepsilon^2 \int_{\Omega_2^\varepsilon} |\nabla v^\varepsilon|^2 dx$ have the same order of magnitude and thus leading to a balance in potential energies. Equations (2.1a) and (2.1b) express the conservation of mass of fluid with Darcy's law in Ω_1^ε and Ω_2^ε respectively. The first equation describes the flow in the fissured regions with large permeability and the second describes the flow in the block system regions with low permeability. For more details, we refer the reader to Arbogast, Douglas, and Hornung [5] (see also Allaire [3]). Condition (2.1c) expresses flux continuity across Γ^ε, the interfacial flow thin layer with permeability given by h^ε. Condition (2.1d) models the imperfect contact between the block and the fissures [8, 9]. The condition (2.1e) is the standard homogeneous Dirichlet condition on the exterior boundary of Ω.

To set the mathematical framework of our Porblem (2.1a)-(2.1e), we need to introduce the following space $H^{\varepsilon} = \left(H^1(\Omega_1^{\varepsilon}) \cap H_0^1(\Omega)\right) \times H^1(\Omega_2^{\varepsilon})$. The space H^{ε} is equipped with the norm

$$\|(\varphi,\psi)\|_{H^{\varepsilon}}^2 = \|\nabla\varphi\|_{L^2(\Omega_1^{\varepsilon})}^2 + \varepsilon^2 \|\nabla\psi\|_{L^2(\Omega_2^{\varepsilon})}^2 + \varepsilon \|\varphi-\psi\|_{L^2(\Gamma^{\varepsilon})}^2 .$$

The weak formulation of (2.1a)-(2.1e) can be read as follows: find $(u^{\varepsilon}, v^{\varepsilon}) \in H^{\varepsilon}$, such that for all $v = (\varphi,\psi) \in H^{\varepsilon}$, we have

$$\begin{aligned} &\int_{\Omega_1^{\varepsilon}} A^{\varepsilon}\nabla v^{\varepsilon}\nabla\varphi dx + \varepsilon^2 \int_{\Omega_2^{\varepsilon}} B^{\varepsilon}\nabla v^{\varepsilon}\nabla\psi dx + \\ &\int_{\Gamma^{\varepsilon}} h^{\varepsilon}(v^{\varepsilon}-v^{\varepsilon})(\varphi-\psi)\, ds^{\varepsilon}(x) = \int_{\Omega_1^{\varepsilon}} f_1\varphi dx + \int_{\Omega_2^{\varepsilon}} f_2\psi dx. \end{aligned} \tag{2.2}$$

where dx and $ds^{\varepsilon}(x)$ denote respectively the Lebesgue measure on $\mathbb{R}^N$ and the Hausdorff measure on Γ^{ε}.

In view of the assumptions made on A_i, f_i and h, we can easily establish the following existence and uniqueness result whose proof is a slight modification of that given by H. I. Ene and D. Polisevski [9] and therefore will be omitted.

Theorem 2.1. *For any sufficiently small $\varepsilon > 0$, there exists a unique couple $(u^{\varepsilon}, v^{\varepsilon}) \in H^{\varepsilon}$, solution of the weak problem (2.2), such that*

$$\| (u^{\varepsilon}, v^{\varepsilon}) \|_{H^{\varepsilon}} \leq C. \tag{2.3}$$

Now, thanks to the a priori estimates (2.3), one is led to study the limiting behavior of the sequence $(u^{\varepsilon}, v^{\varepsilon})$ as ε approaches 0. This is summarized in the main result of the paper:

Theorem 2.2. *Let $(u^{\varepsilon}, v^{\varepsilon}) \in H^{\varepsilon}$ be the solution of the weak system (2.2). Then, up to a subsequence, there exists $u \in H_0^1(\Omega)$ and $v_0 \in L^2\left(\Omega; H_{\#}^1(Y_2)\right)$ such that*

$$\begin{aligned} u^{\varepsilon} &\rightharpoonup u \text{ in } L^2(\Omega) \text{ weakly,} \\ v^{\varepsilon} &\rightharpoonup \int_{Y_2} v_0(y)\, dy \text{ in } L^2(\Omega) \text{ weakly,} \end{aligned}$$

Let $w^{\varepsilon} = \chi_1\left(\frac{x}{\varepsilon}\right) u^{\varepsilon} + \chi_2\left(\frac{x}{\varepsilon}\right) v^{\varepsilon}$ denote the overall pressure. Then, w^{ε} weakly converges to $U = u + G$ where U is the unique solution to the homogenized model:

$$\begin{aligned} &-\operatorname{div}\left(A^{\mathrm{h}}\nabla U\right) = F \text{ in } \Omega, \\ &U = G \text{ on } \partial\Omega. \end{aligned}$$

Here, A^{h} (resp. F, G) is given by (3.11) (resp. (3.14),(3.16)).

To prove this theorem, we shall first use the two-scale asymptotic procedure to formally derive the homogenized model and apply then the two-scale convergence technique to rigorously justify the homogenization. This is the scope of the next two Sections.

3. The formal homogenization procedure

The purpose of this section is to formally construct the homogenized system of (2.1a)-(2.1e), via the two-scale asymptotic expansion method [6, 10]. Assume the following ansatz:

$$u^{\varepsilon}(x) = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \dots, \tag{3.1}$$

$$v^{\varepsilon}(x) = v_0(x,y) + \varepsilon v_1(x,y) + \varepsilon^2 v_2(x,y) + \dots \tag{3.2}$$

where $y = x/\varepsilon$, the unknowns $u_0,\ v_0,\ u_1, v_1,\ u_2, v_2...$ are Y-periodic in the second variable y. Plugging the above expansions (3.1)-(3.2) into the set of equations (2.1a–2.1e) and identifying powers of ε yields a hierarchy of boundary value problems. At the first step, it is not difficult to observe that Equation (2.1b) at ε^{-2}, and Equation (2.1c) at ε^{-1} orders yield that u_0 is independent of $y \in Y_1$. That is

$$u_0(x,y) = u(x).$$

Next, Equation (2.1b) at ε^{-1} and Equation (2.1c) at ε^0 orders show that the corrector term u_1 may be written as

$$u_1(x,y) = \sum_{j=1}^{N} \frac{\partial u}{\partial x_j}(x)\,\omega_j(y) + \tilde{u}(x) \tag{3.3}$$

where, for $1 \le j \le N$, $\omega_j \in H^1_\#(Y_1)/\mathbb{R}$ is the unique solution to the following cell problem:

$$\int_{Y_1} (A\nabla_y \omega_j, \nabla_y \zeta)\, dy = \int_{Y_1} (-Ae_j, \nabla_y \zeta)\, dy,\ \zeta \in H^1_\#(Y_1),$$

where (e_j) is the canonical basis of $\mathbb{R}^N$. In (3.3), $\tilde{u}(x)$ is any additive constant. At the final step, Equation (2.1a)-(2.1b) at ε^0 and Equations (2.1c)-(2.1d)) at ε^1 orders yield the system:

$$-\operatorname{div}_y(A\nabla_y u_2) = f_1 + \operatorname{div}_y(A\nabla_x u_1) + \operatorname{div}_x(A(\nabla_y u_1 + \nabla_x u)) \text{ in } \Omega \times Y_1, \tag{3.4}$$

$$-\operatorname{div}_y(B\nabla_y v_0) = f_2 \text{ in } \Omega \times Y_2; \tag{3.5}$$

$$A\nabla_y u_2 \cdot \nu = -A\nabla_x u_1 \cdot \nu + B\nabla_y v_0 \cdot \nu \text{ on } \Omega \times \Gamma, \tag{3.6}$$

$$A\nabla_y u_2 \cdot \nu = -A\nabla_x u_1 \cdot \nu - h(u - v_0) \text{ on } \Omega \times \Gamma, \tag{3.7}$$

$$y \mapsto v_0(x,y),\ u_2(x,y)\ Y - \text{periodic}. \tag{3.8}$$

The weak formulation of Equations (3.4), (3.7)-(3.8) is

$$\int_{Y_1} (A\nabla_y u_2, \nabla_y \zeta)\, dy = \langle F, \zeta \rangle,\quad \zeta \in H^1_\#(Y_1),$$

where

$$\begin{aligned} \langle F, \zeta \rangle &= \int_{Y_1} \operatorname{div}_x(A(\nabla_y u_1 + \nabla u))\,\zeta \\ &\quad - \int_{Y_1} A\nabla_x u_1 \nabla_y \zeta + \int_\Gamma h(u - v_0)\,\zeta ds(y) + \int_{Y_1} f_1 \zeta \end{aligned}$$

where $ds(y)$ denotes the Hausdorff measure on Γ. Again, using the Divergence Theorem (as in [10]), a necessary and sufficient condition for the existence and uniqueness of $u_2 \in H^1_\#(Y_1)/\mathbb{R}$ is that F satisfies the compatibility condition $\langle F, 1 \rangle = 0$. This reads in our case as follows:

$$\int_{Y_1} (\operatorname{div}_x(A(\nabla_y u_1 + \nabla u)))\, dy + \int_\Gamma h(u - v_0)\, ds(y) = \int_{Y_1} f_1 dy. \tag{3.9}$$

Using (3.3), Equation (3.9) becomes

$$-\operatorname{div}\left(A^{\mathrm{h}} \nabla u\right) + \int_\Gamma h(u - v_0)\, ds(y) = \int_{Y_1} f_1 dy \tag{3.10}$$

where $A^{\mathrm{h}} = \left(a^{\mathrm{h}}_{ij}\right)_{1 \le i,j \le N}$ is given by

$$a^{\mathrm{h}}_{ij} = \int_{Y_1} A(\nabla_y \omega_i + e_i) \cdot (\nabla_y \omega_j + e_j)\, dy. \tag{3.11}$$

On the other hand, it is easy to see that v_0 can be written as

$$v_0(x,y) - u(x) = \alpha(y) f_2(x), \quad (x,y) \in \Omega \times Y_2 \tag{3.12}$$

where $\alpha \in H^1_{\#}(Y_2)$ is the unique solution of the following problem

$$\begin{cases} -\mathrm{div}_y(B\nabla_y \alpha) = 1 \text{ in } Y_2, \\ B\nabla_y \alpha \cdot \nu - h\alpha = 0 \text{ on } \Gamma. \end{cases} \tag{3.13}$$

Therefore, Equation (3.10) yields

$$-\operatorname{div}\left(A^{\mathrm{h}}\nabla u\right) = |Y_1| f_1 + \hat{\alpha} f_2 := F^* \tag{3.14}$$

where $\hat{\alpha} = \int_\Gamma \alpha ds(y)$. The boundary condition for u is obtained from (2.1e) at ε^0 order and it reads

$$u = 0 \text{ on } \partial\Omega. \tag{3.15}$$

Let us observe that the overall pressure $w^\varepsilon = \chi_1\left(\frac{x}{\varepsilon}\right) u^\varepsilon + \chi_2\left(\frac{x}{\varepsilon}\right) v^\varepsilon$ two scale converges to $u + \chi_2 \alpha f_2$ (see Definition 4.1 below). Consequently, w^ε weakly converges to $U = u + G$ where

$$G = \left(\int_{Y_2} \alpha\right) f_2. \tag{3.16}$$

Equation (3.14) is the so-called macroscopic equation for u. Moreover, if G is sufficiently smooth, then the homogenized model for the weak limit U is as follows:

$$-\operatorname{div}\left(A^{\mathrm{h}}\nabla U\right) = F \text{ in } \Omega, \tag{3.17}$$

$$U = G \text{ on } \partial\Omega \tag{3.18}$$

where $F = F^* + \operatorname{div}\left(A^{\mathrm{h}}\nabla G\right)$. Note that in the homogenized problem (3.17)-(3.18), a non homogeneous Dirichlet is derived, on the contrary to the micro-model, where homogeneous boundary condition is prescribed, see condition (2.1e). This extra source surface density essentially arises from the fact that

(1) Blocks have low permeability;
(2) non null source volumetric density on the blocks;
(3) and the contribution of the fluid flow in the vicinity of the thin layer where the contact is assumed imperfect.

4. Two-scale convergence approach

In this section, we will derive the homogenized system of (2.1a)-(2.1e) by the two scale convergence method, see [3]. First, we define $C_{\#}(Y)$ to be the space of all continuous functions on $\mathbb{R}^3$ which are Y-periodic. Let the space $L^2_{\#}(Y)$ (resp. $L^2_{\#}(Y_i)$, $i = 1,2$) to be all functions belonging to $L^2_{\mathrm{loc}}\left(\mathbb{R}^3\right)$ (resp. $L^2_{\mathrm{loc}}(Z_i)$) which are Y-periodic, and $H^1_{\#}(Y)$ (resp. $H^1_{\#}(Y_i)$) to be the space of those functions together with their derivatives belonging to $L^2_{\#}(Y)$ (resp. $L^2_{\#}(Z_i)$). Now, we recall the definition and main results concerning the method of two-scale convergence. For full details, we refer the reader to [3, 4].

Definition 4.1. *A sequence (v^ε) in $L^2(\Omega)$ two-scale converges to $v \in L^2(\Omega \times Y)$ (we write $v^\varepsilon \xrightarrow{2-s} v$) if, for any admissible test function $\varphi \in L^2(\Omega; \mathcal{C}_{\#}(Y))$,*

$$\lim_{\varepsilon \to 0} \int_\Omega v^\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega \times Y} v(x,y) \varphi(x,y)\, dxdy.$$

Theorem 4.1. *Let (v^ε) be a sequence of functions in $L^2(\Omega)$ which is uniformly bounded. Then, there exist $v \in L^2(\Omega \times Y)$ and a subsequence of (v^ε) which two-scale converges to v.*

Theorem 4.2. *Let (v^ε) be a uniformly bounded sequence in $H^1(\Omega)$ (resp. $H_0^1(\Omega)$). Then there exist $v \in H^1(\Omega)$ (resp. $H_0^1(\Omega)$) and $\hat{v} \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$ such that, up to a subsequence,*

$$v^\varepsilon \overset{2-s}{\rightharpoonup} v; \qquad \nabla v^\varepsilon \overset{2-s}{\rightharpoonup} \nabla v + \nabla_y \hat{v}.$$

Theorem 4.3. *Let (v^ε) be a sequence of functions in $H^1(\Omega)$ such that*

$$\|v^\varepsilon\|_{L^2(\Omega)} + \varepsilon \|\nabla v^\varepsilon\|_{L^2(\Omega)^3} \leq C.$$

Then, there exist $v \in L^2\left(\Omega; H^1_\#(Y)\right)$ and a subsequence of (v^ε), still denoted by (v^ε) such that

$$v^\varepsilon \overset{2-s}{\rightharpoonup} v, \qquad \varepsilon \nabla v^\varepsilon \overset{2-s}{\rightharpoonup} \nabla_y v$$

and for every $\varphi \in \mathcal{D}(\Omega; \mathcal{C}_\#(Y))$, we have:

$$\lim_{\varepsilon \to 0} \int_{\Gamma^\varepsilon} \varepsilon v^\varepsilon(x)\, \varphi\left(x, \frac{x}{\varepsilon}\right) ds^\varepsilon(x) = \int_{\Omega \times \Gamma} v(x,y)\, \varphi(x,y)\, dx ds(y).$$

Here and in the sequel $ds(y)$ denotes the Hausdorff measure on Γ.

As a direct application of the theorems listed above (Thms 4.1-4.3) and the a priori estimates (2.3), we give without proof the following two-scale convergence result concerning the solutions $(u^\varepsilon, v^\varepsilon)$ of the Problem (2.2).

Theorem 4.4. *There exists a subsequence of $(u^\varepsilon, v^\varepsilon)$, solution of (2.2), still denoted $(u^\varepsilon, v^\varepsilon)$, and there exist unique $u \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$ and $v_0 \in L^2\left(\Omega; H^1_\#(Y_2)\right)$ such that*

$$\chi_1^\varepsilon u^\varepsilon \overset{2-s}{\rightharpoonup} \chi_1 u, \;\; \chi_2^\varepsilon v^\varepsilon \overset{2-s}{\rightharpoonup} \chi_2 v_0 \tag{4.1}$$

and

$$\chi_1^\varepsilon \nabla u^\varepsilon \overset{2-s}{\rightharpoonup} \chi_1 (\nabla u + \nabla_y u_1), \;\; \varepsilon \chi_2^\varepsilon \nabla v^\varepsilon \overset{2-s}{\rightharpoonup} \chi_2 \nabla_y v_0. \tag{4.2}$$

Moreover, the following convergence holds:

$$\lim_{\varepsilon \to 0} \int_{\Gamma^\varepsilon} \varepsilon (u^\varepsilon - v^\varepsilon)\, \psi^\varepsilon ds^\varepsilon(x) = \int_{\Omega \times \Gamma} (u - v_0)\, \psi dx ds(y), \tag{4.3}$$

for any $\psi \in \mathcal{D}(\Omega; \mathcal{C}_\#(Y))$ with $\psi^\varepsilon(x) = \psi(x, x/\varepsilon)$.

To determine the limiting equations of the system (2.2), we begin by choosing the adequate admissible test functions. Let $\varphi^\varepsilon(x) = \varphi(x) + \varepsilon \varphi_1\left(x, \frac{x}{\varepsilon}\right)$ and $\psi^\varepsilon(x) = \psi\left(x, \frac{x}{\varepsilon}\right)$ where $\varphi \in \mathcal{D}(\Omega)$ and $\varphi_1, \psi \in \mathcal{D}\left(\Omega; \mathcal{C}^\infty_\#(Y)\right)$. Taking $\varphi = \varphi^\varepsilon$ and $\psi = \psi^\varepsilon$ in (2.2), we obtain

$$\begin{aligned} &\int_{\Omega_1^\varepsilon} A\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon (\nabla \varphi + \nabla_y \varphi_1)\, dx + \int_{Q_2^\varepsilon} \varepsilon B\left(\frac{x}{\varepsilon}\right) \nabla v^\varepsilon \nabla_y \psi dx + \\ &\varepsilon \int_{\Gamma^\varepsilon} h\left(\frac{x}{\varepsilon}\right) (u^\varepsilon - v^\varepsilon)(\varphi - \psi)\, ds^\varepsilon(x) + \varepsilon R^\varepsilon = \int_{\Omega_1^\varepsilon} f_1 \varphi dx + \int_{\Omega_2^\varepsilon} f_2 \psi dx \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} R^\varepsilon = &\int_{\Omega_1^\varepsilon} A\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon \nabla_x \varphi_1\left(x, \frac{x}{\varepsilon}\right) dx + \varepsilon \int_{\Omega_2^\varepsilon} B\left(\frac{x}{\varepsilon}\right) \nabla v^\varepsilon \nabla_x \psi\left(x, \frac{x}{\varepsilon}\right) dx + \\ &\varepsilon \int_{\Gamma^\varepsilon} h\left(\frac{x}{\varepsilon}\right) (u^\varepsilon - v^\varepsilon)\, \varphi_1\left(x, \frac{x}{\varepsilon}\right) ds^\varepsilon(x). \end{aligned}$$

In view of (4.2), one can deduce that

$$\lim_{\varepsilon\to 0}\int_{\Omega_1^\varepsilon} A\left(\frac{x}{\varepsilon}\right)\nabla v^\varepsilon\left(\nabla\varphi(x)+\nabla_y\varphi_1\left(x,\frac{x}{\varepsilon}\right)\right)dx =$$
$$\int_{\Omega\times Y}\chi_1(y)\,A(y)\left(\nabla u+\nabla_y u_1\right)\left(\nabla\varphi(x)+\nabla_y\varphi_1(x,y)\right)dxdy$$

and

$$\lim_{\varepsilon\to 0}\int_{\Omega_2^\varepsilon}\varepsilon B\left(\frac{x}{\varepsilon}\right)\nabla v^\varepsilon\nabla_y\psi\left(x,\frac{x}{\varepsilon}\right)dx =$$
$$\int_{\Omega\times Y}\chi_2(y)\,B(y)\,\nabla v_0\nabla_y\psi(x,y)\,dxdy.$$

By virtue of (4.3), we find that

$$\lim_{\varepsilon\to 0}\varepsilon\int_{\Gamma^\varepsilon} h\left(\frac{x}{\varepsilon}\right)\left(u^\varepsilon-v^\varepsilon\right)\left(\varphi(x)-\psi\left(x,\frac{x}{\varepsilon}\right)\right)ds^\varepsilon(x) =$$
$$\int_{\Omega\times\Gamma} h(y)\left(u-v_0\right)\left(\varphi(x)-\psi(x,y)\right)dxds(y)\,.$$

We observe that $R^\varepsilon = O(1)$ and, by collecting all the preceeding limits, we get the following limiting equation of (2.2):

$$\int_{\Omega\times Y_1} A(y)\left(\nabla u+\nabla_y u_1\right)\left(\nabla\varphi+\nabla_y\varphi_1\right)dxdy+\int_{\Omega\times Y_2} B(y)\,\nabla_y v_0\nabla_y\psi dxdy+$$
$$\int_{\Omega\times\Gamma} h(y)\left(u-v_0\right)\left(\varphi-\psi\right)dxds(y)$$
$$\int_{\Omega\times Y_1} f_1\varphi dx+\int_{\Omega\times Y_2} f_2\psi dx. \tag{$\cancel{4.5}$}$$

By density argument, the equation (4.5) still holds true for any $(\varphi,\varphi_1,\varphi_2)\in H_0^1(\Omega)\times L^2\left(\Omega;H_\#^1(Y_1)/\mathbb{R}\right)\times L^2\left(\Omega;H_\#^1(Y_2)\right)$. We can summarize the preceding by observing that these equations are a weak formulation associated to the two-scale homogenized system (4.6)-(4.12) below:

$$-\mathrm{div}_y\left(A\left(\nabla u+\nabla_y u_1\right)\right)=0 \text{ a.e. in } \Omega\times Y_1, \tag{4.6}$$
$$-\mathrm{div}_y\left(B\nabla_y v_0\right)=f_2 \text{ a.e.in } \Omega\times Y_2, \tag{4.7}$$
$$-\mathrm{div}\left(\int_{Y_1} A\left(\nabla u+\nabla_y u_1\right)dy\right)+$$
$$\int_\Gamma h(y)\left[u-v_0\right]ds(y)=f_1 \text{ a.e. in } \Omega, \tag{4.8}$$

with the transmission and boundary conditions:

$$\left(A\left(\nabla u+\nabla_y u_1\right)\right)\cdot\nu=0 \text{ a.e. on } \Omega\times\Gamma, \tag{4.9}$$
$$B\nabla_y v_0\cdot v=-h(y)\left[u-v_0\right] \text{ a.e. on } \Omega\times\Gamma, \tag{4.10}$$
$$y\longmapsto u_1,v_0\ Y\text{-periodic}, \tag{4.11}$$
$$u=0 \text{ on } \partial\Omega. \tag{4.12}$$

Let us first note that Equations (4.6) and (4.9) lead to the relation (3.3). Similarly, Equations (4.7), (4.10) and (4.11) yield (3.12). Equation (4.8) is the same as (3.9). With these remarks in mind, one can easily recover the homogenization procedure of the preceeding section and thus we have rigorously justified the formal two-scale asymptotic expansion method.

5. Conclusion

We have used the homogenization theory to derive a macro-model for fluid flow in fissured media with microstructures, in which blocks present very low permeabilities. Moreover, the contact between the block region and the fissures region is of imperfect type. The micro-model that we considered in this paper is with a homogeneous Dirichlet boundary condition prescribed on the exterior boundary. We have shown that the overall behavior of fluid flow in such heterogeneous media with low permeability at the micro-scale is a classical problem except that in the vicinity of the surface, there is an additional source density arising from the source density of the blocks at the micro-scale and the fact that the contact at micro-scale is imperfect.

References

[1] A. Ainouz, Homogenized double porosity models for poro-elastic media with interfacial flow barrier. Mathematica Bohemica **136** (2011), 357–365.

[2] A. Ainouz, Homogenization of a Double Porosity Model in Deformable Media, *to appear in Electr. J. of Diff. Equ.*

[3] G. Allaire, Homogenization and two scale convergence, SIAM J. of Math. Anal. **23** (1992), 1482–1519.

[4] G. Allaire, A. Damlamian & U. Hornung, Two scale convergence on periodic surfaces and applications. A. Bourgeat et *al.* eds., 15–25, World Scientific Pub., Singapore (1996).

[5] T. Arbogast, J. Douglas & U. Hornung, Derivation of the double porosity model of single phase flow via homogenization theory, SIAM J. Math. Anal. **21** (1990) pp. 823-836.

[6] A. Bensoussan, J. L. Lions & G. Papanicolaou, Asymptotic analysis for periodic structures North Holland, Amsterdam, 1978.

[7] G. W. Clark, R. E. Showalter, Two-scale convergence for a flow in a partially fissured medium, Electr. J. of Diff. Equ., **1999** (1999) pp. 1-20.

[8] H. Deresiewicz & R. Skalak, On uniqueness in dynamic poroelasicity. Bull. Seismol. Soc. Amer., **53** (1963) pp 783–788.

[9] H.I. Ene, D. Polisevski,

[10] E. Sanchez-Palencia, Non-Homogeneous Media and Vibration Theory, Lecture Notes in Physics **127**, 1980.

Laboratory AMNEDP, Faculty of Mathematics, University of sciences and technology, Po Box 32 El Alia, Algiers, Algeria.

E-mail address: aainouz@usthb.dz